Gurobi

Mixed-Integer Nonlinear Optimization in Gurobi 13

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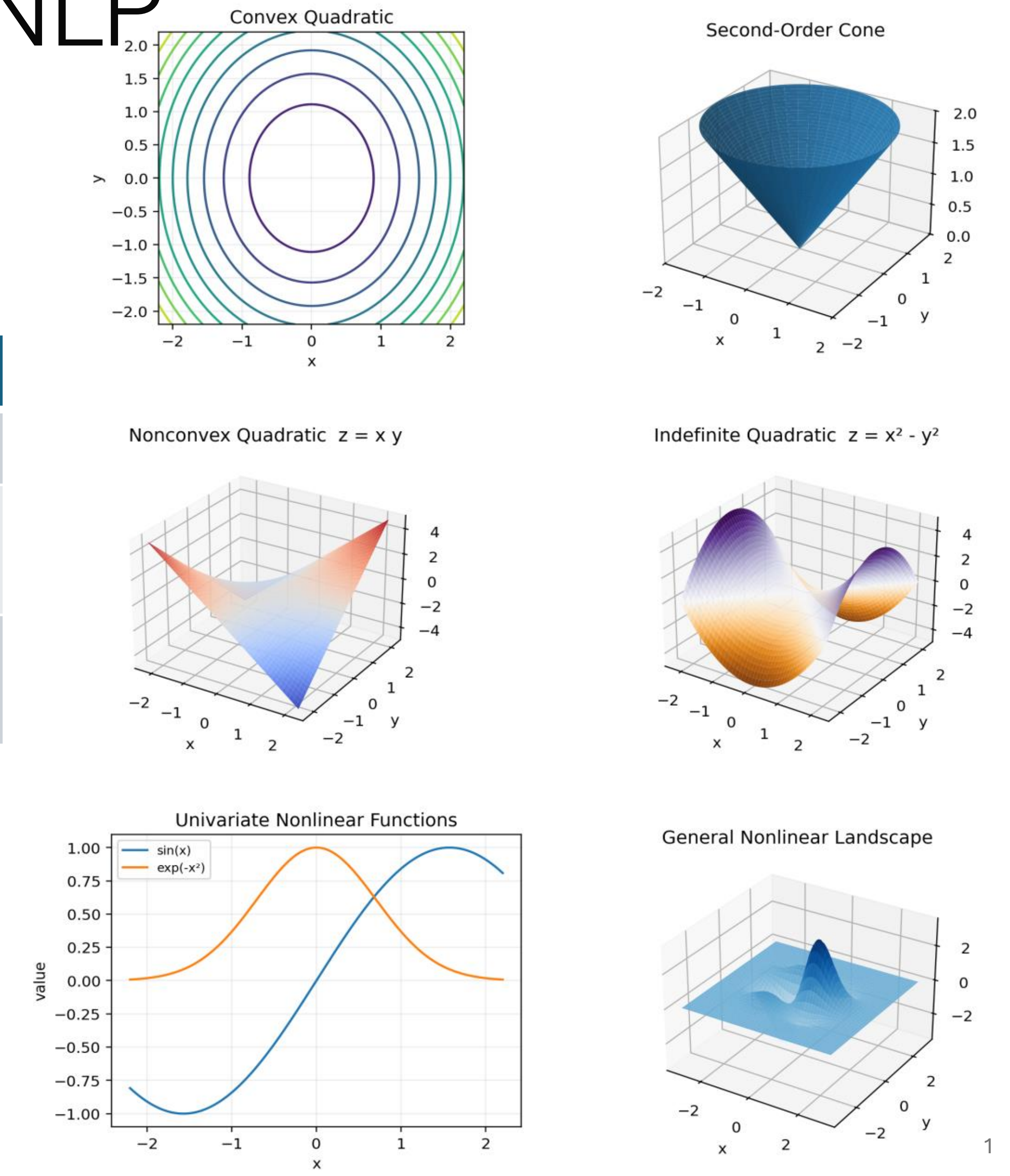
From Convex Quadratic to MINLP

Version	Class	
4.0	Quadratic Objective	QP, MIQP
5.0	**Convex** Quadratic Constraints (SOC)	QCP, SOCP, MIQCP
9.0-13.0	Nonconvex quadratic and more	NLP, MINLP



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Sources of Nonlinearity

Convex quadratic

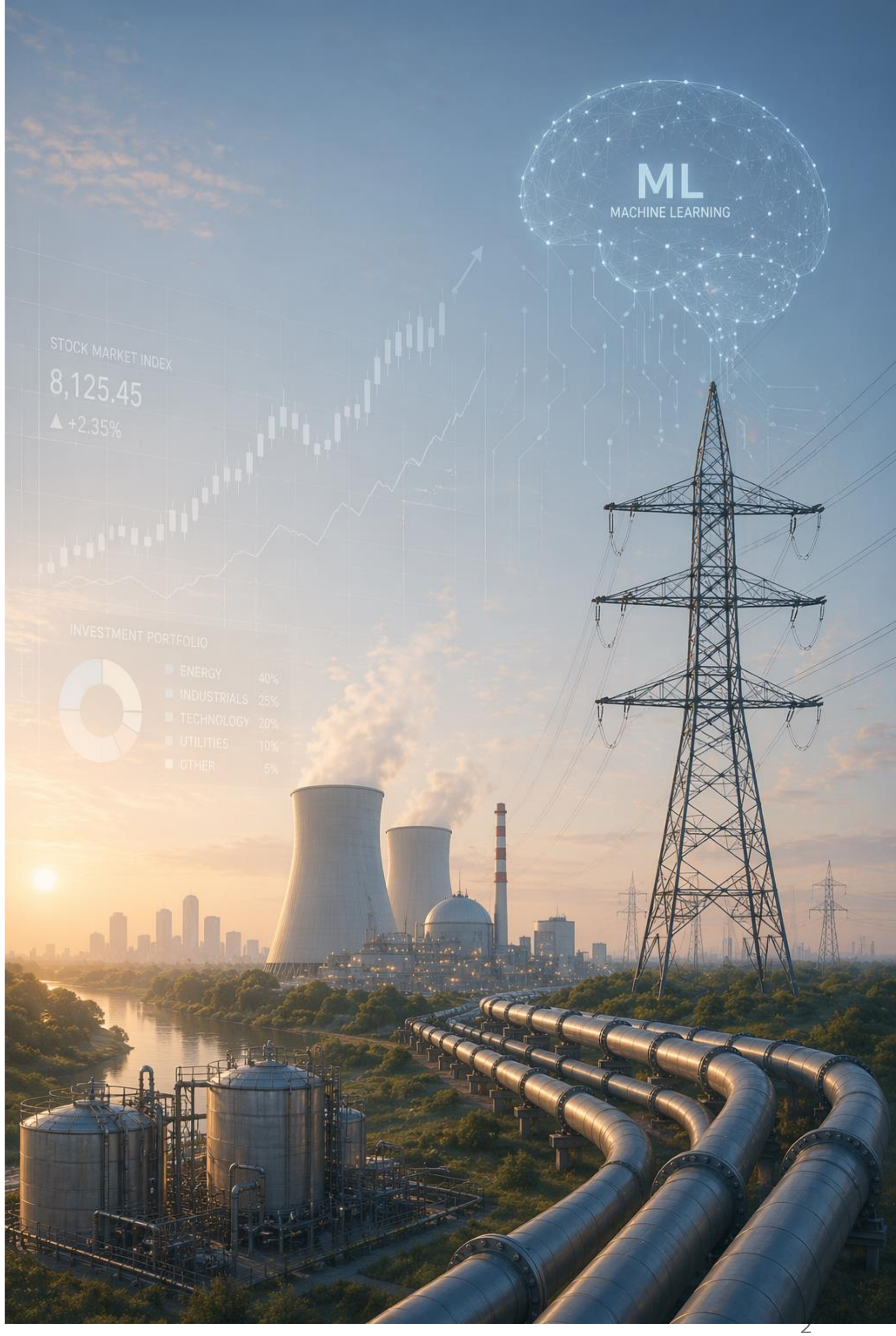
- risk (finance), delays (networks), physical forces

Nonconvex quadratic

- blending, pooling (petro-chemical, agriculture)

General nonlinear

- trigonometric (ACOPF)
- sign-power (gas and liquid networks)
- tanh, logistic (ML activations)



The MINLP Goal

$$
\begin{aligned}
&\min f(x) \\
&\text{s.t.} \\
&g_i(x) \le 0 && i = 1, \ldots m, \\
&x_j \in \mathbb{Z} && j = 1, \ldots, p, \\
&l \le x \le u, x \in \mathbb{R}^n.
\end{aligned}
$$

(f and g_i algebraic and sufficiently smooth)

Fundamental Hardness

- Contrary to MILP the continuous relaxation may not be *easy* to solve
- Unbounded variables lead to **undecidable** MINLPs
- Integral variables (bounded) can be written as nonlinear constraints:

$$(x - 0) \times (x - 1) \times (x - 2) = 0 \leftrightarrow x \in \{0,1,2\}$$

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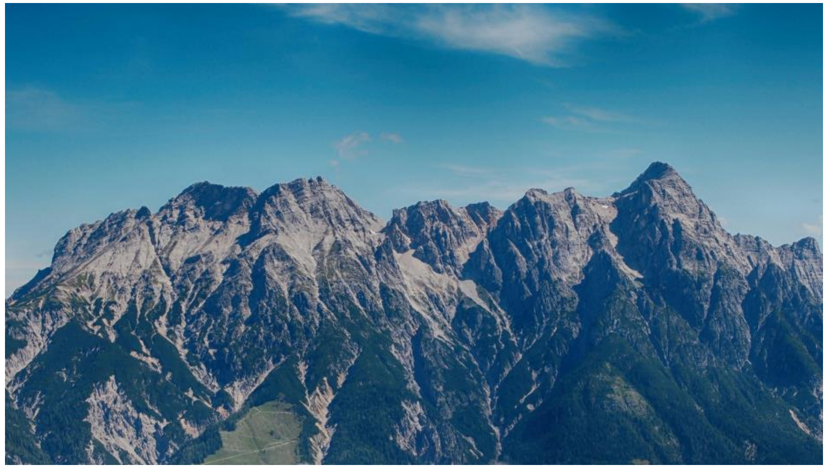
Convex vs. Nonconvex

- Finding the highest point
  - Start at some point
  - Take improving steps inside feasible region



- Finding the highest point
  - Start at some point
  - Take improving steps inside feasible region
  - Converges only to locally optimal solution

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Agenda

MIQPs and MIQCPs (convex)

MINLPs (nonconvex)

Conclusions

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Convex MIQPs and MIQCPs

Agenda

Definition and Formulation

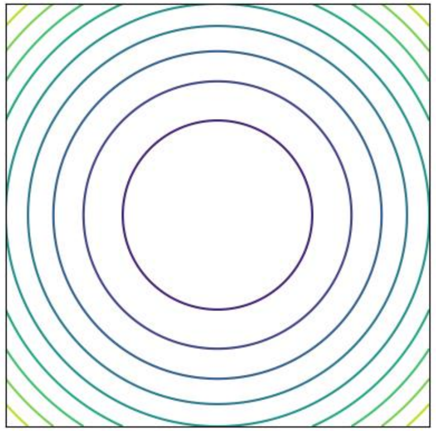
Algorithms Sketch

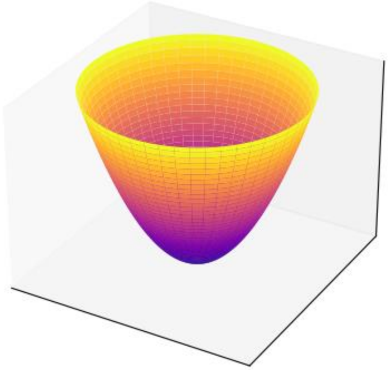
Example

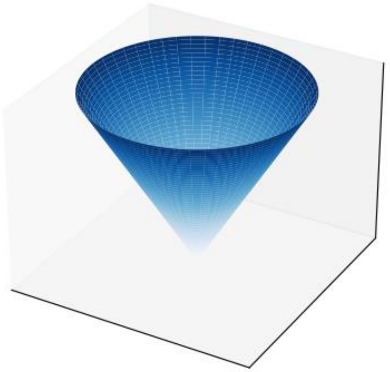
Numerical Results

Convex MIQP/MIQCP

$$
\begin{aligned}
&\min \quad c^T x + x^T Q^0 x \\
&\text{s.t:} \\
&a_i x + x^T Q^i x \le b_i, \; i = 1, \ldots, m, \\
&x_j \in \mathbb{Z} \qquad\qquad\qquad \mathrm{j} = 1, \ldots, \mathrm{p}, \\
&\mathrm{l} \le \mathrm{x} \le \mathrm{u}
\end{aligned}
$$







- $Q^0 \succcurlyeq 0$ is positive semidefinite
- $a_i x + x^T Q^i x \le b_i$ are *second-order cone representable*
- Continuous relaxation *easy* to solve (simplex if only objective, barrier with constraints)

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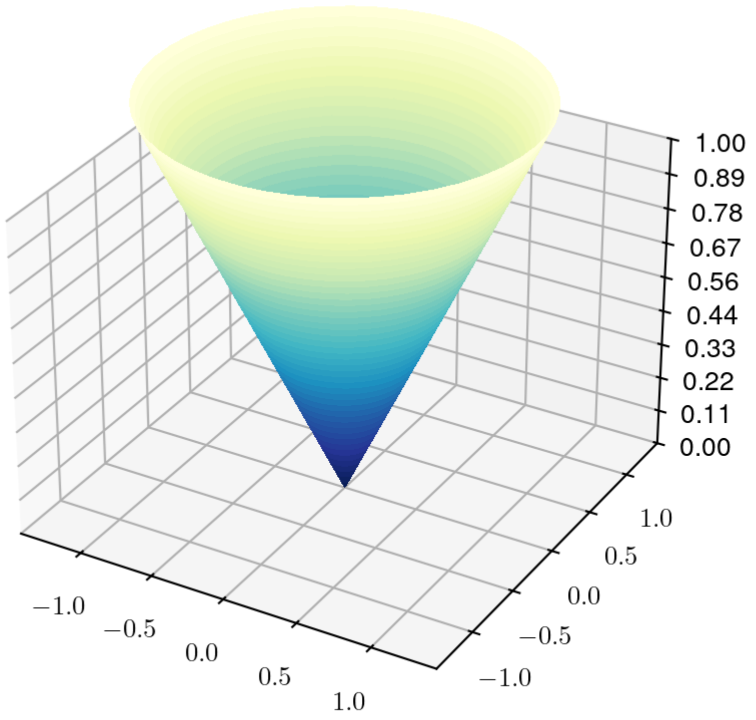
Second-Order Cone



$$\mathcal{L}^n = \{x \in \mathbb{R}^{n+1}: \sum_{j=1}^{n} x_j^2 \leq x_0^2, x_0 \geq 0\}$$

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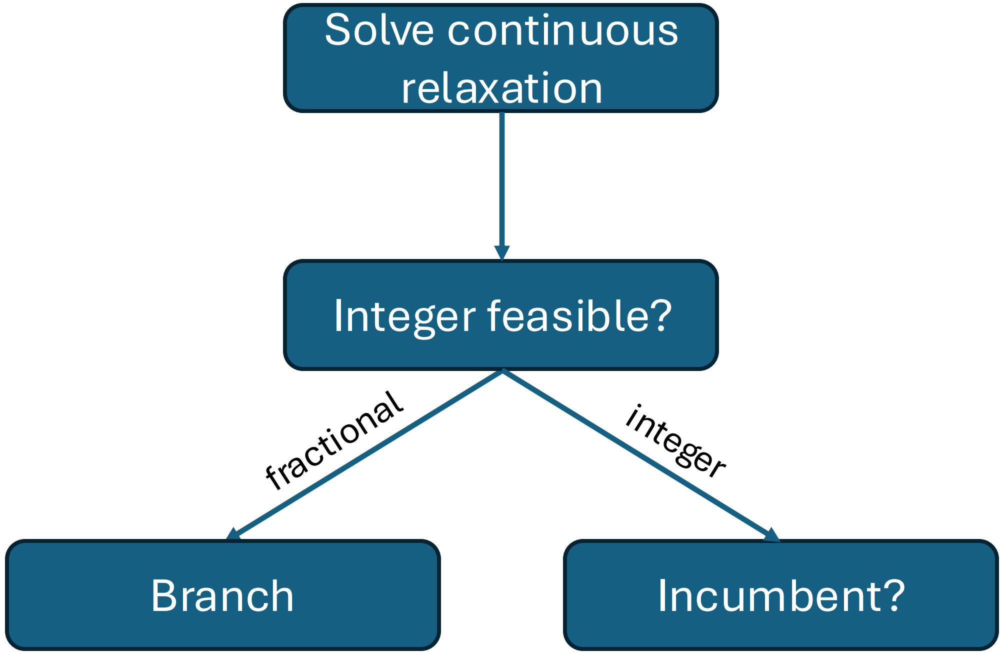
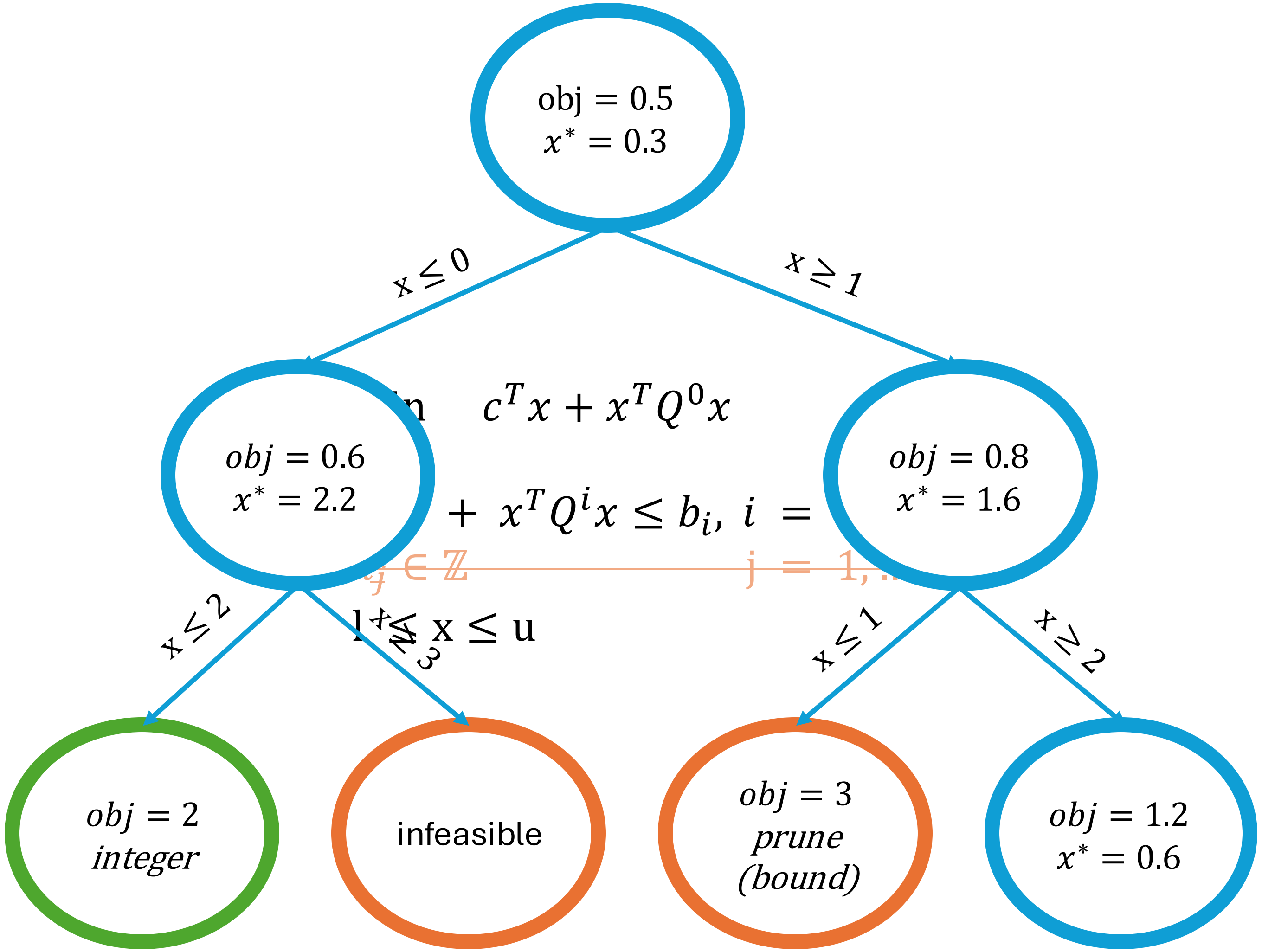
Second-Order Cone Representability

- $\sum_{i=1}^{n} x_i^2 \leq x_0^2$, with $x_0 \geq 0$
- $\sum_{i=2}^{n} x_i^2 \leq x_0 x_1$, with $x_0, x_1 \geq 0$ (rotated SOC)
- $a^T x + x^T Q x \leq b$, with $Q \succcurlyeq 0$
- $x^T Q x \leq y^2$, with $Q \succcurlyeq 0, y \geq 0$

- Very powerful but modeling sometimes far from obvious
- SOC barrier to solve it

Branch-and-Bound





integer nodes

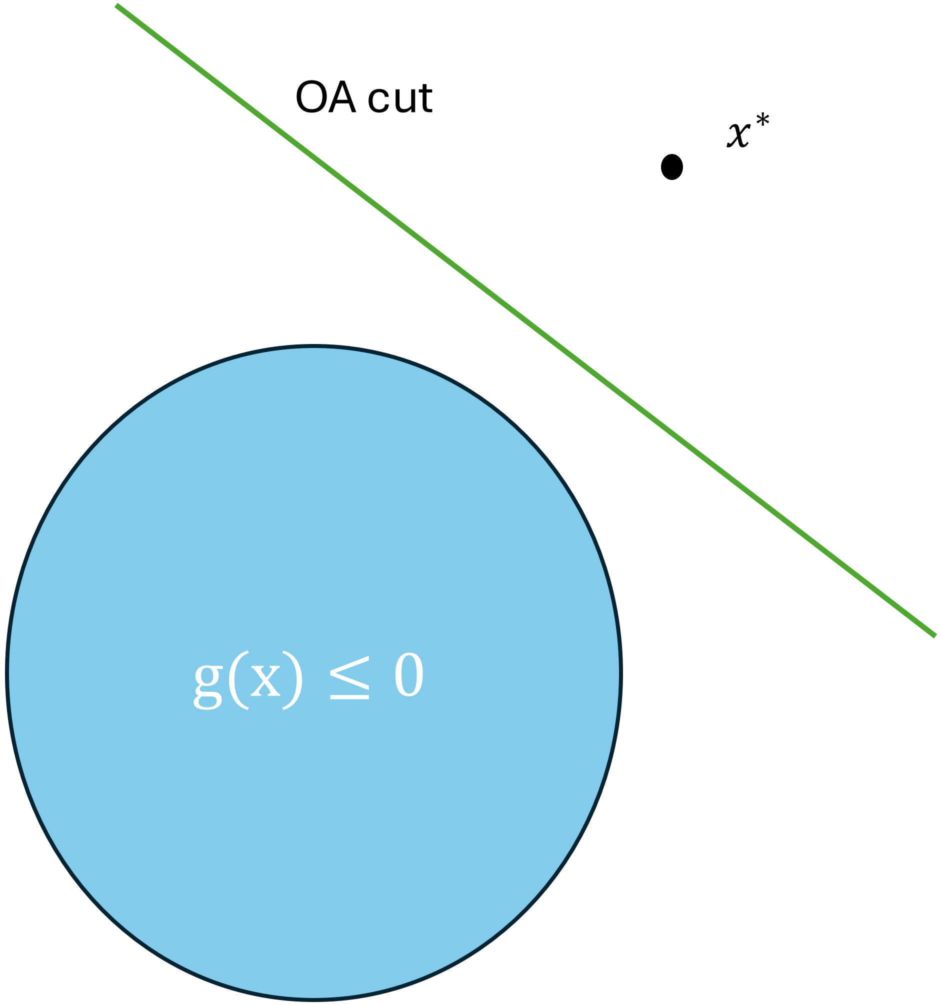
pruned

Outer Approximation

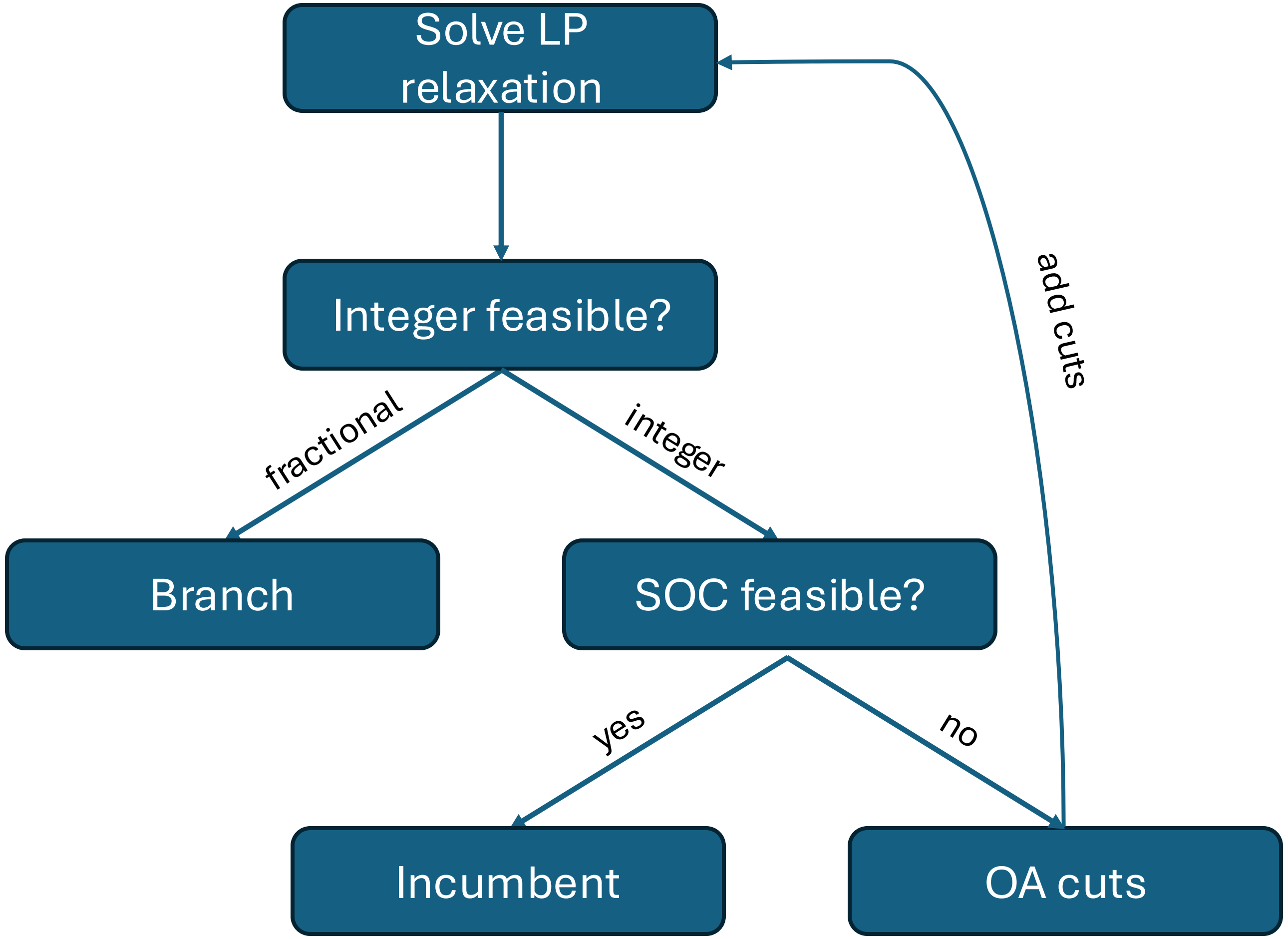
- Convex constraint $g(x) \leq 0$
- For any $x^* \in \mathbb{R}^n$, build linear approximation:

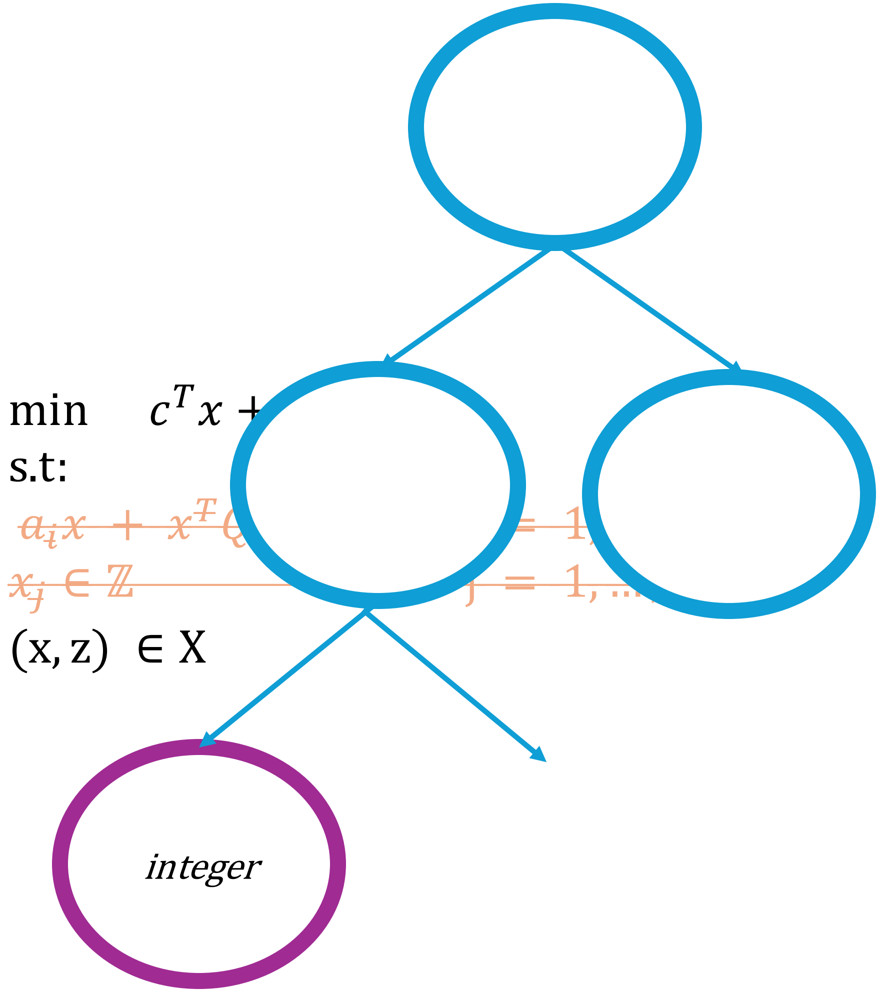
$$\nabla g(x^*)(x - x^*) + g(x^*) \leq 0$$

- A valid cut, if $g(x^*) > 0$ removes x^*

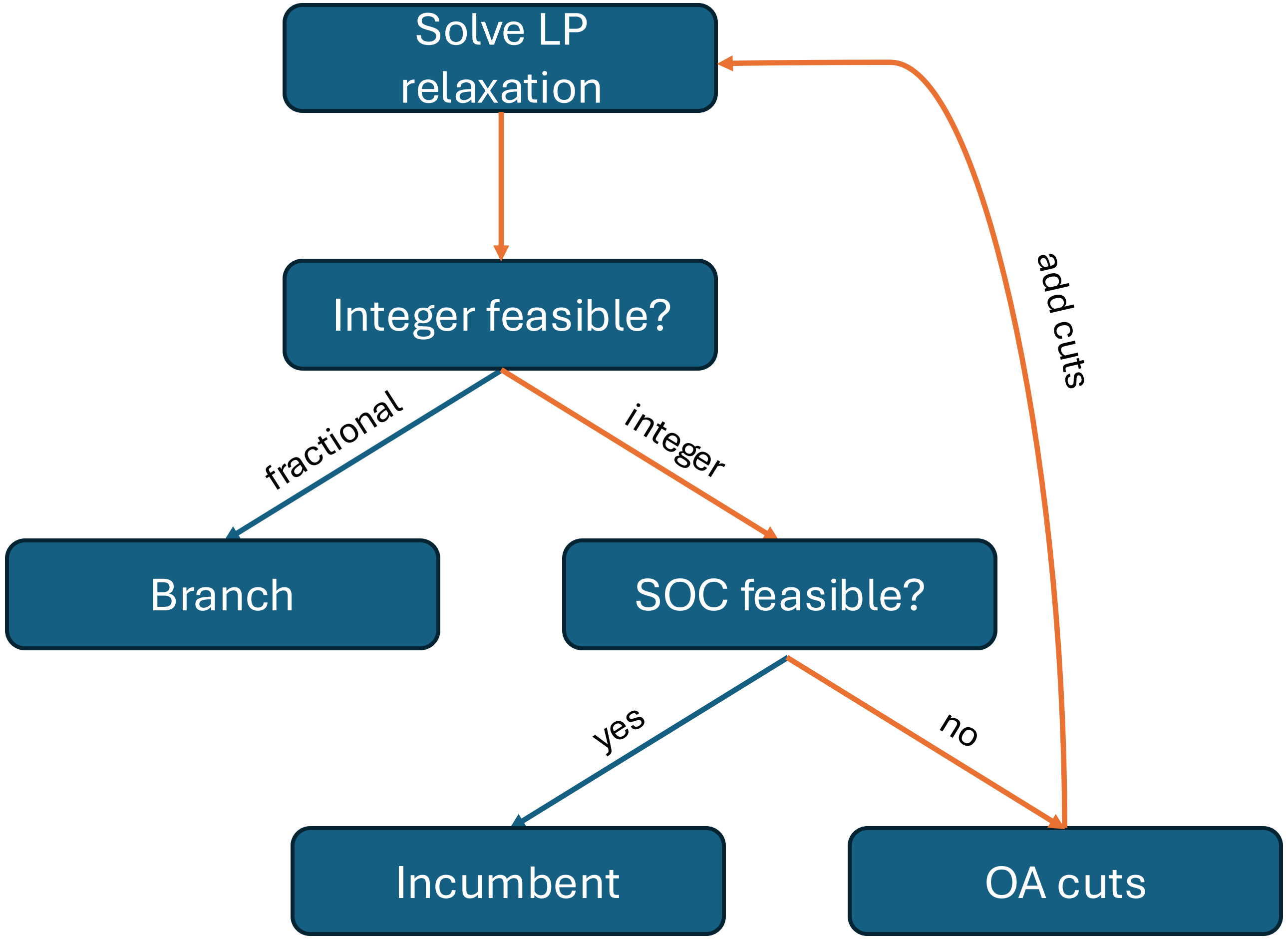


OA Branch-and-Cut





Pitfalls of OA Based Approach



- Because of tolerances, it may happen:
  - Solution of LP relaxation integer
  - Doesn't satisfy some quadratic constraint to ϵ
  - Linear cuts are not deep enough for changing LP solution

- Dilemma
  - Accept a violated solution
  - Report a sub-optimal status
  - **Try with barrier to resolve the node**

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Formulating an MIQCP with gurobipy

Cardinality constrained portfolio optimization

- Maximize expected return
- Control risk using a variance limit
  - Quadratic constraint
- Select at most *K* assets
  - Combinatorial

```
m = gp.Model()
x = m.addMVar(n, name="x")
z = m.addMVar(n, vtype="B", name="z")

m.setObjective(μ @ x, gp.GRB.MAXIMIZE)
m.addConstr(x @ Σ @ x <= max_risk)
m.addConstr(x.sum() == 1.0)

m.addConstr(x <= z)
m.addConstr(z.sum() <= K)
```

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Solving a 462-Asset Portfolio

Data from gurobi-finance[1]: n=462, K=40, max_risk=3

```
     0     0    0.31821    0    -          -    0.31821      -      -   11s
*    0     0               0       0.3182129    0.31821  0.00%      -   11s

Explored 1 nodes (6851 simplex iterations) in 11.16 seconds (7.83 work units)
Thread count was 11 (of 11 available processors)

Solution count 1: 0.318213
No other solutions better than 0.318213

Optimal solution found (tolerance 1.00e-04)
Best objective 3.182129040457e-01, best bound 3.182129040457e-01, gap 0.0000%
```

1: https://gurobi-finance.readthedocs.io/

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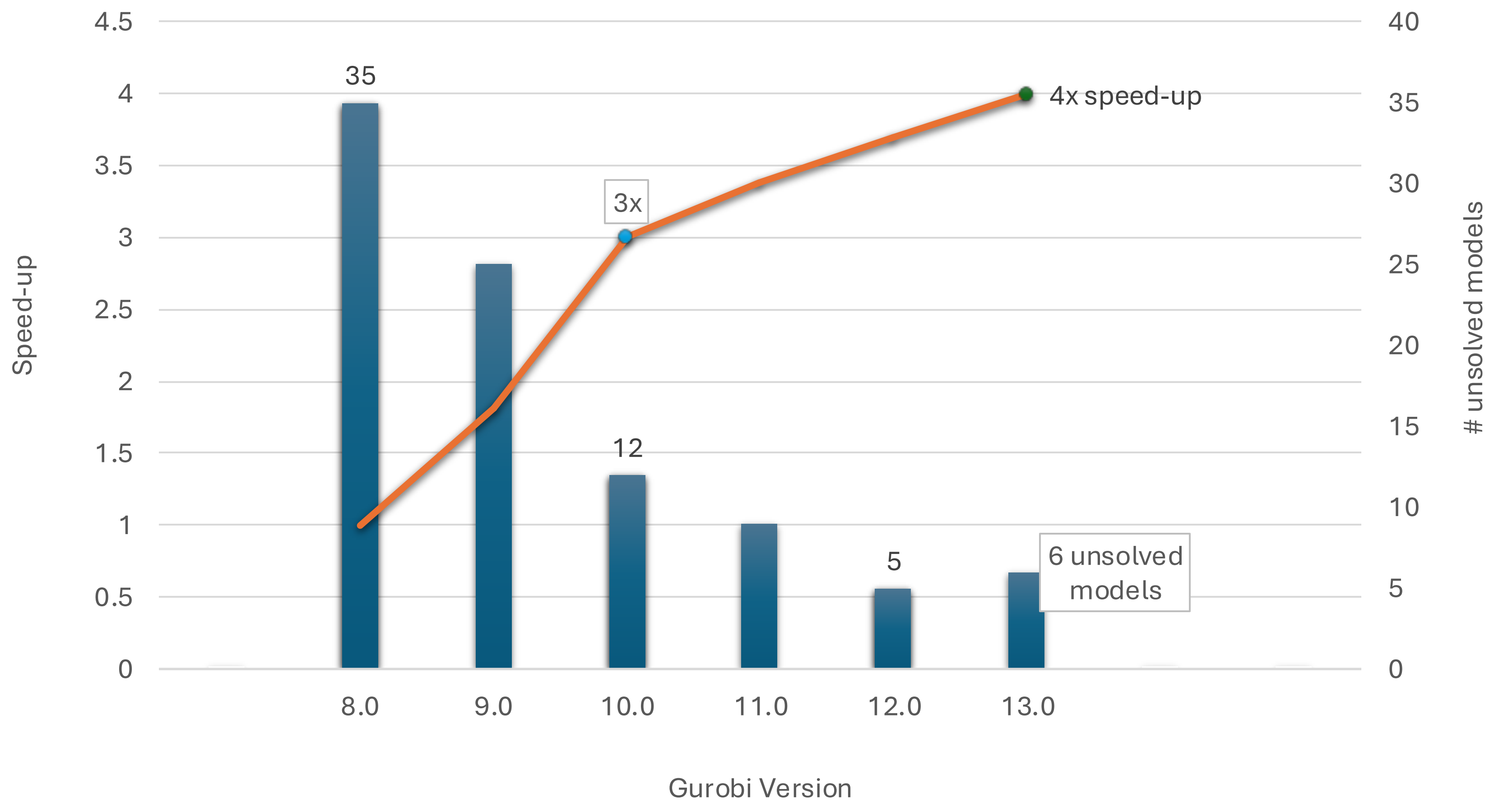
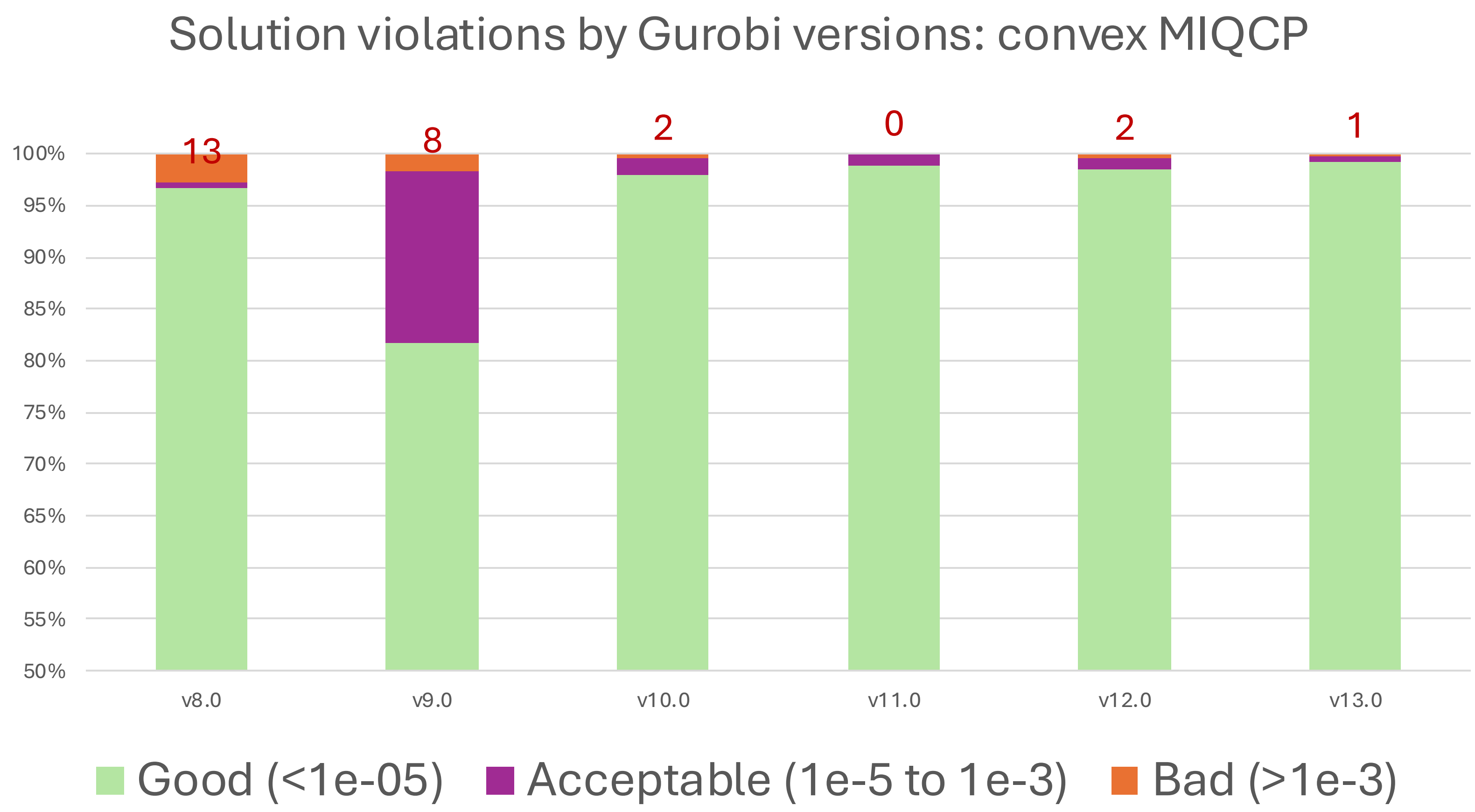
Improving MIQCP Performance



- Internal benchmark (convex MIQCPs)
- 41 models not solved by any solver
- Gurobi 8.0: 2018
- Steady improvement

Improving MIQCP Solution Quality



- Small glitch in Gurobi 9.0
- Very good since Gurobi 10.0

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Issues can arise easily with non-default

Same data from gurobi-finance: n=462, K=40, max_risk=3

```
Set parameter MIQCPMethod to value 1
Set parameter PreMIQCPForm to value 1



    0     0    0.31821    0    -    0.31624    0.31821  0.62%     -   18s

Explored 1 nodes (7100 simplex iterations) in 18.89 seconds (37.28 work units)
Thread count was 11 (of 11 available processors)

Solution count 3: 0.316238 0.314297 0.31348

Sub-optimal termination (unable to solve some node relaxations)
Best objective 3.182129486537e-01, best bound 3.182129644216e-01, gap 0.62%
```

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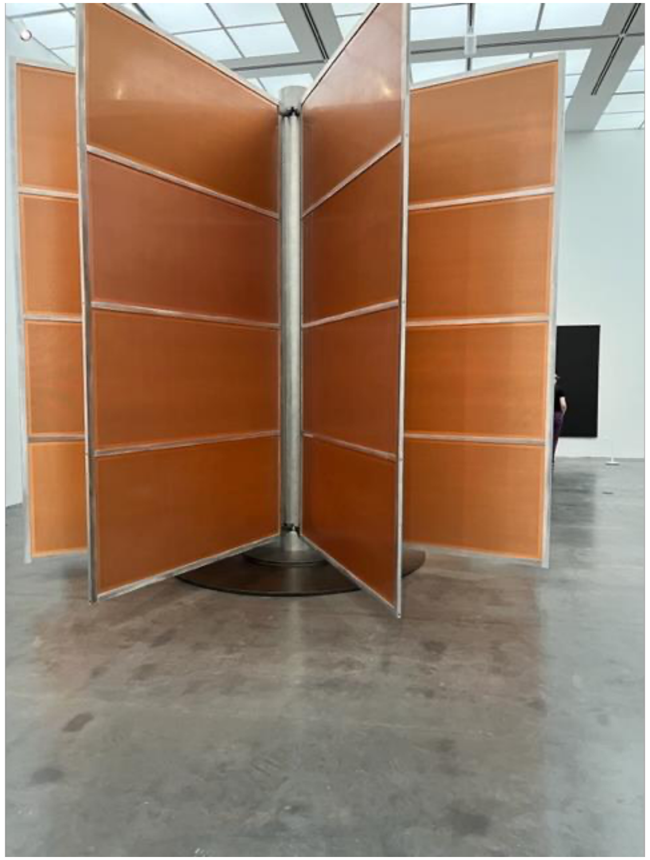
Nonconvex MINLP



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Nonconvex Quadratic

General MINLP

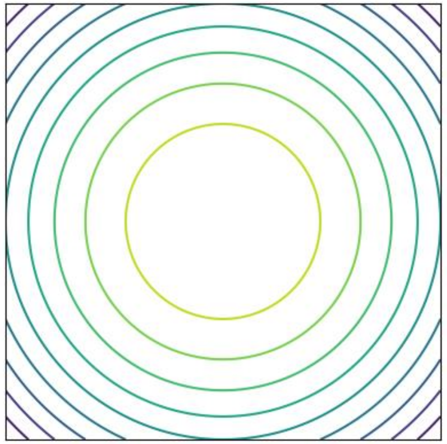
Example

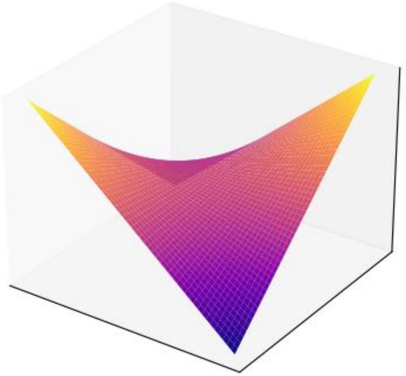
Numerical Results

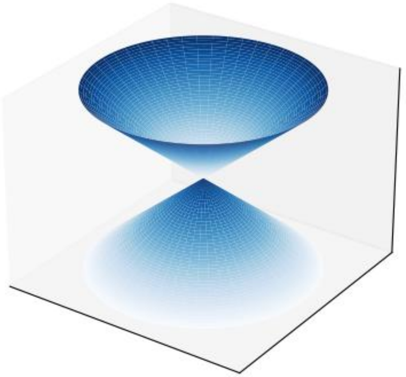
Nonconvex MIQP/MIQCP (Gurobi ≥ 9)

$$\begin{aligned}
\min \quad & c^T x + x^T Q^0 x \\
\text{s.t:} \quad & \\
& a_i x + x^T Q^i x \le b_i, \quad i = 1, \ldots, m, \\
& x_j \in \mathbb{Z} \qquad\qquad\qquad j = 1, \ldots, p, \\
& l \le x \le u
\end{aligned}$$

- $Q^0, Q^1, \ldots, Q^m$ symmetric (w.l.o.g.)
- Continuous relaxation *hard* to solve







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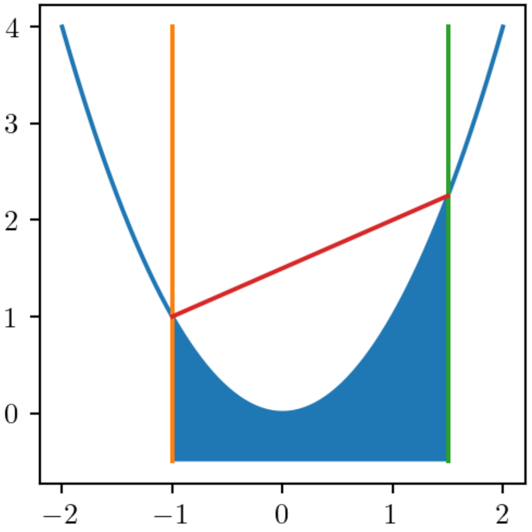
Bilinear Reformulation and Relaxation

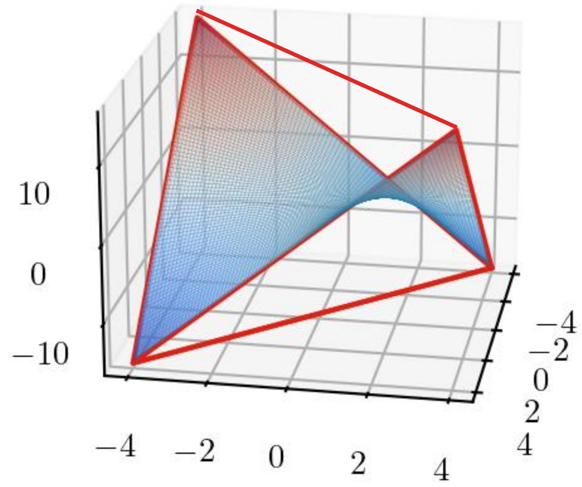
- For each product $x_i x_j$
  - Introduce a new variable z_{ij}
  - Add the **bilinear** term $z_{ij} = x_i x_j$
  - Replace product with z_{ij}
  - Relax nonconvex constraint with **convex envelopes**

$$
\begin{aligned}
&\min c^T x + \langle Q^0, Z \rangle \\
&s.t. \\
&a_k^T x + \langle Q^k, Z \rangle \leq b_k, k = 1, \ldots, m \\
&z^-(x_i, x_j) \leq z_{ij} \leq z^+(x_i, x_j), \\
&l \leq x \leq u
\end{aligned}
$$

$$(\langle Q, Z \rangle = \sum_i \sum_j q_{ij} z_{ij})$$

Convex Envelopes





$$z_{ii}^{-} = x_i^2$$
$$z_{ii}^{+} = (u_i + l_i)x_i - l_i u_i$$

$$z_{ij}^{-} = \max\begin{Bmatrix} l_j x_i + l_i x_j - l_i l_j \\ u_j x_i + u_i x_j - u_i u_j \end{Bmatrix}$$
$$z_{ij}^{+} = \min\begin{Bmatrix} l_j x_i + u_i x_j - u_i l_j \\ u_j x_i + l_i x_j - l_i u_j \end{Bmatrix}$$

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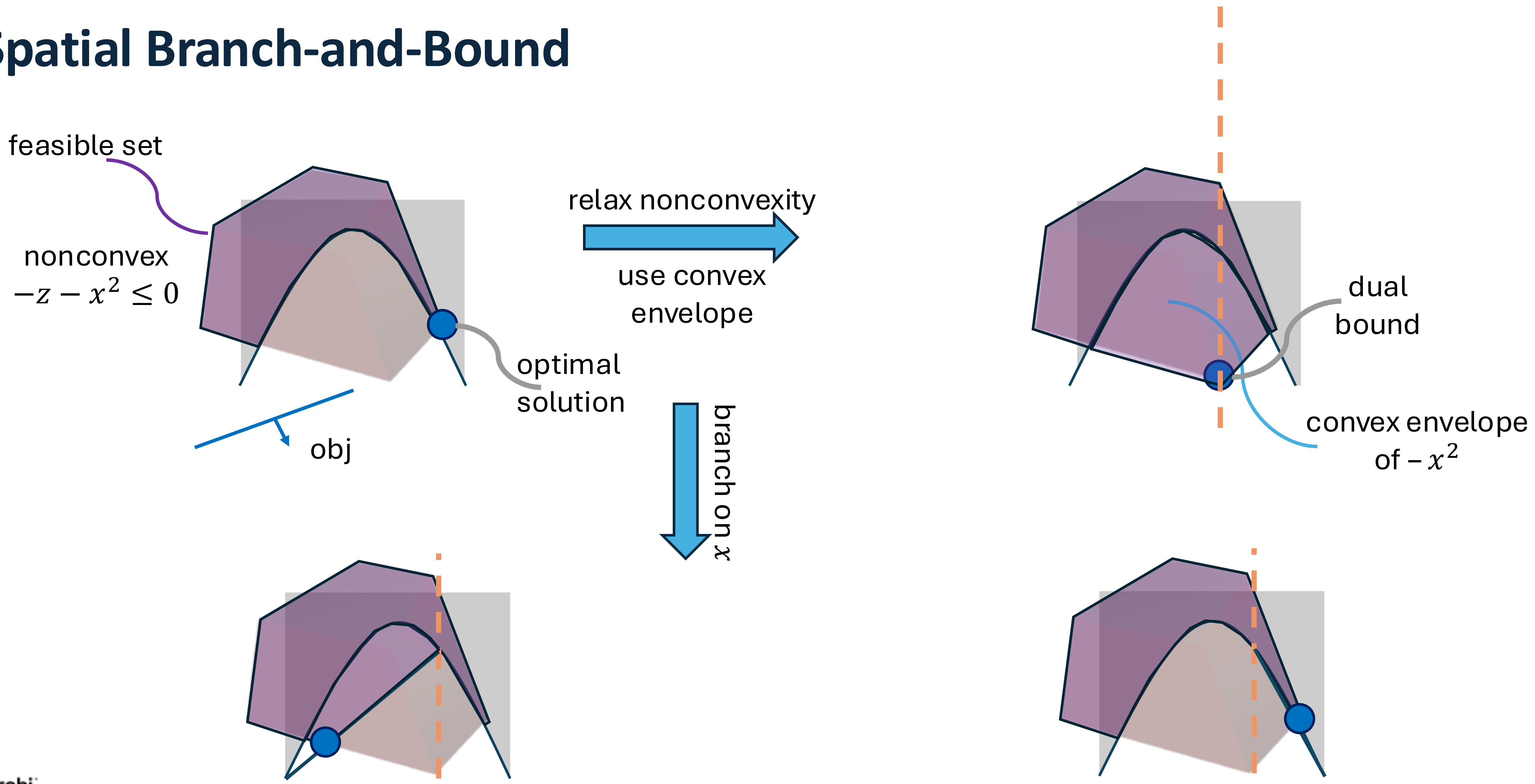
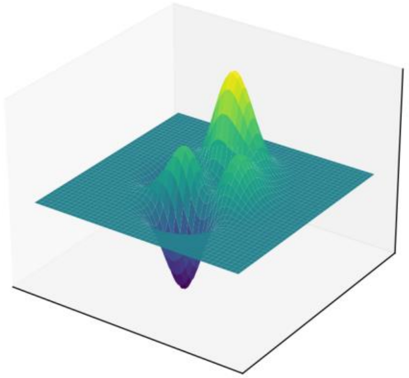
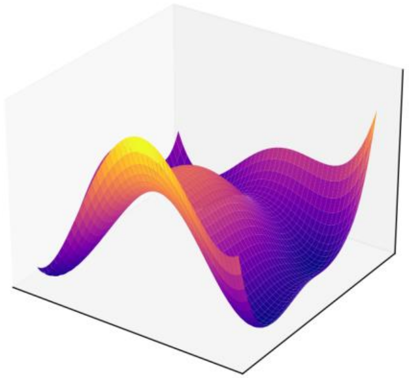
Spatial Branch-and-Bound

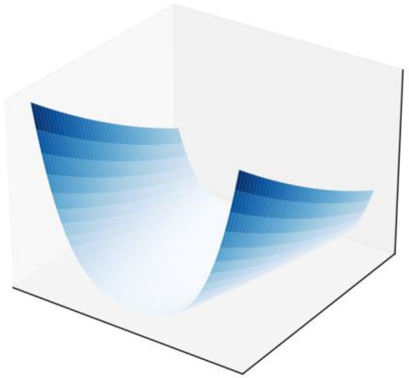


MINLP (Gurobi ≥ 12)

$$
\begin{aligned}
&\min f(x) \\
&\text{s.t.} \\
&g_i(x) \le 0 && i = 1, \dots m, \\
&x_j \in \mathbb{Z} && j = 1, \dots, p, \\
&l \le x \le u, x \in \mathbb{R}^n.
\end{aligned}
$$







- f and g_i are composed of:
  - $+, *, -, /$
  - $x^a, a^x, \exp(x), \log(x), \log_2(x), \log_{10}(x)$
  - $\sin(x), \cos(x), \tan(x)$
  - $\text{logistic}(x), \tanh(x)$
  - $\text{signpow}(x, a) = \text{sign}(x) \cdot |x|^a, a \in \mathbb{R}_{\ge 1}$

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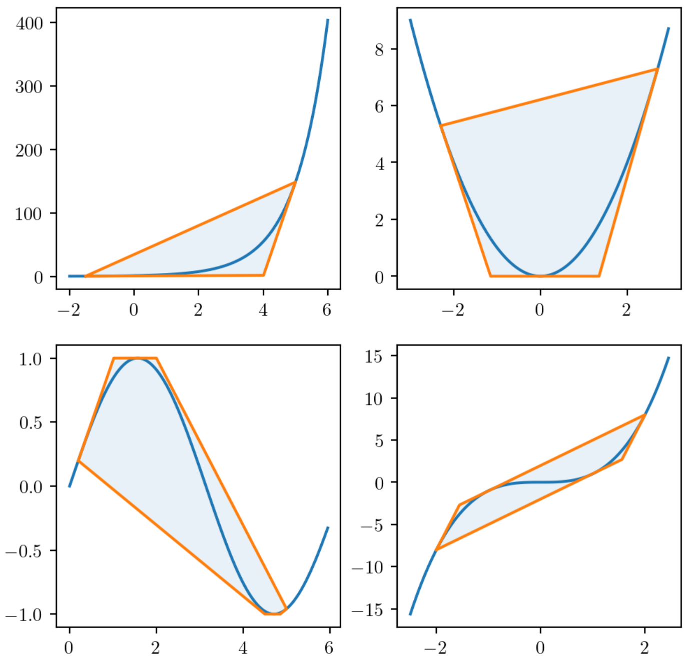
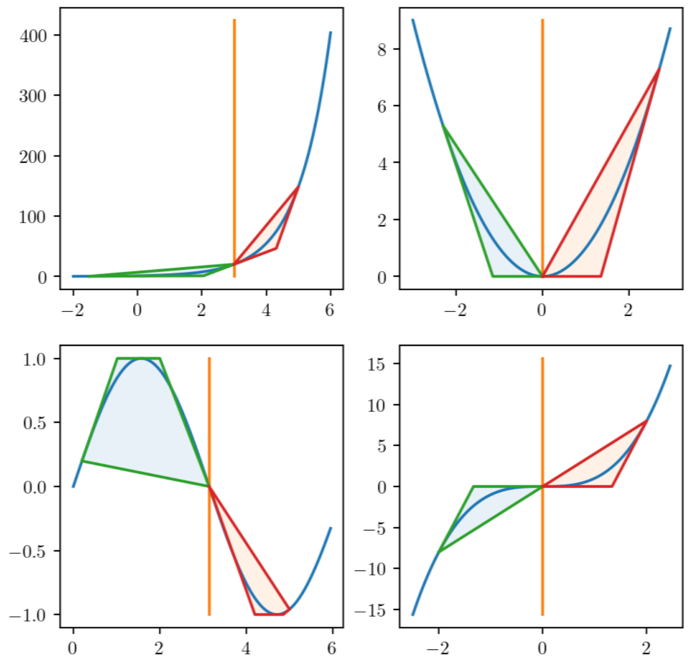
Spatial Branch-and-Bound for MINLP



Branching **improves** the relaxation



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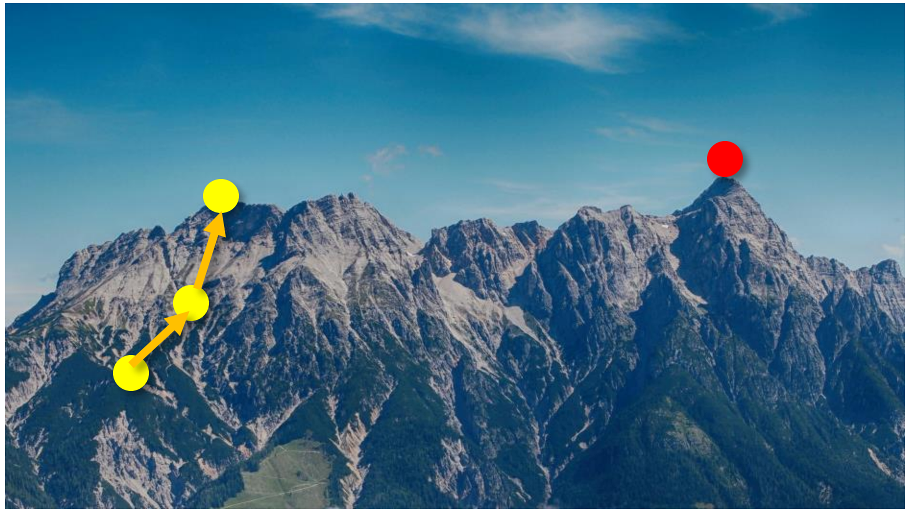
Local vs. Global

NonConvex Region



- Finding the highest point
  - Start at some point
  - Take improving steps inside feasible region
  - Converges only to locally optimal solution
- Need divide-and-conquer search algorithm to find globally optimal solution
  - Combinatorial explosion

- Finding a local optimum is "easier" than proving global optimality
- But it is not easy:
  - Formally NP-hard
  - Problems are large, very large
  - Problems are nonlinear, very nonlinear
- Algorithms to find local optima generally fall into the domain of nonlinear optimization
- Essential to get solutions with high accuracy (i.e. small infeasibility)

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Nonlinear Barrier in Gurobi

Version	Class		NL Barrier
9.0	Nonconvex quadratic	Nonconvex QCP/MIQCP	Introduced in 9.5 as NLPheur heuristic
11.0	Univariate functions	NLP/MINLP	Extend to univariate function
12.0	Nonlinear expressions	NLP/MINLP	
13.0	Continuous models	NLP	**Directly callable**

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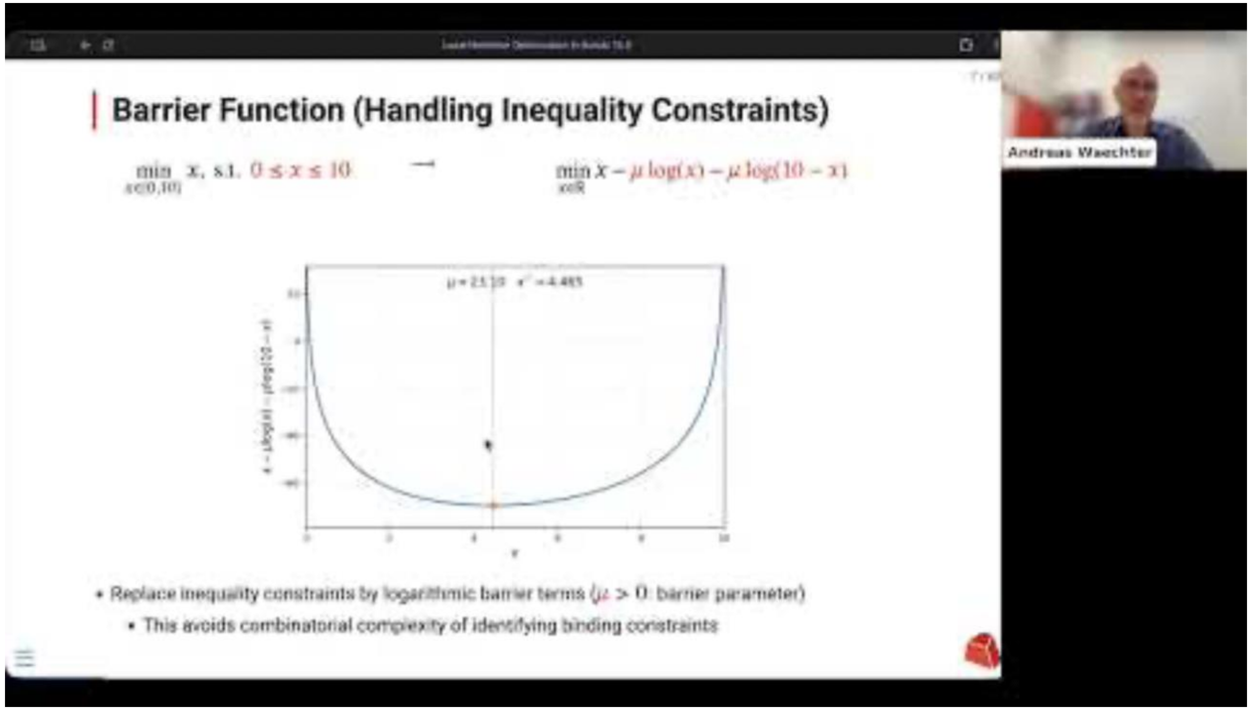
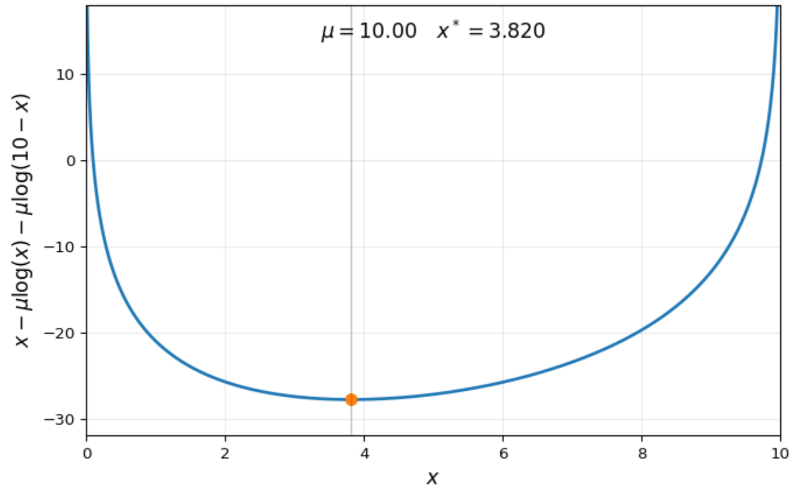
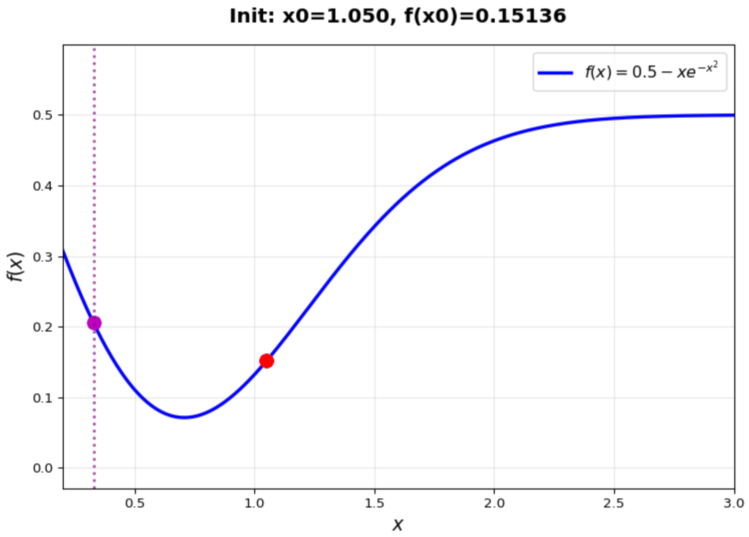
Nonlinear Barrier Algorithm



youtube.com/watch?v=WyHGTAajCpg&feature=youtu.be





NL Barrier: Solve Rates Improvements

Dataset	v12	v13	∇(pts)
Internal Benchmarks			
nlpall	55.0%	90.5%	**+35.5%**
ncqcpall	69.3%	91.2%	**+21.9%**
Academic Benchmarks			
CUTE	54.2%	84.3%	**+30.1%**
pglib (ACOPF)			
ac-rect	41.4%	97.0%	**+55.6%**
ac-polar	0.5%	99.0%	**+98.5%**

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Integration of Barrier in the Global Solver

- NL Barrier used as *"heuristic"* in global solver
- Goal: finding good locally optimal solution
- Controlled by parameter `NLPheur`, **new values:**
  - -1: default
  - 0: off
  - 1: mild (essentially run once or twice at the root)
  - 2: moderate (models with integers, run for every *assignment* encountered)
  - 3: aggressive (run at node with node local bounds)

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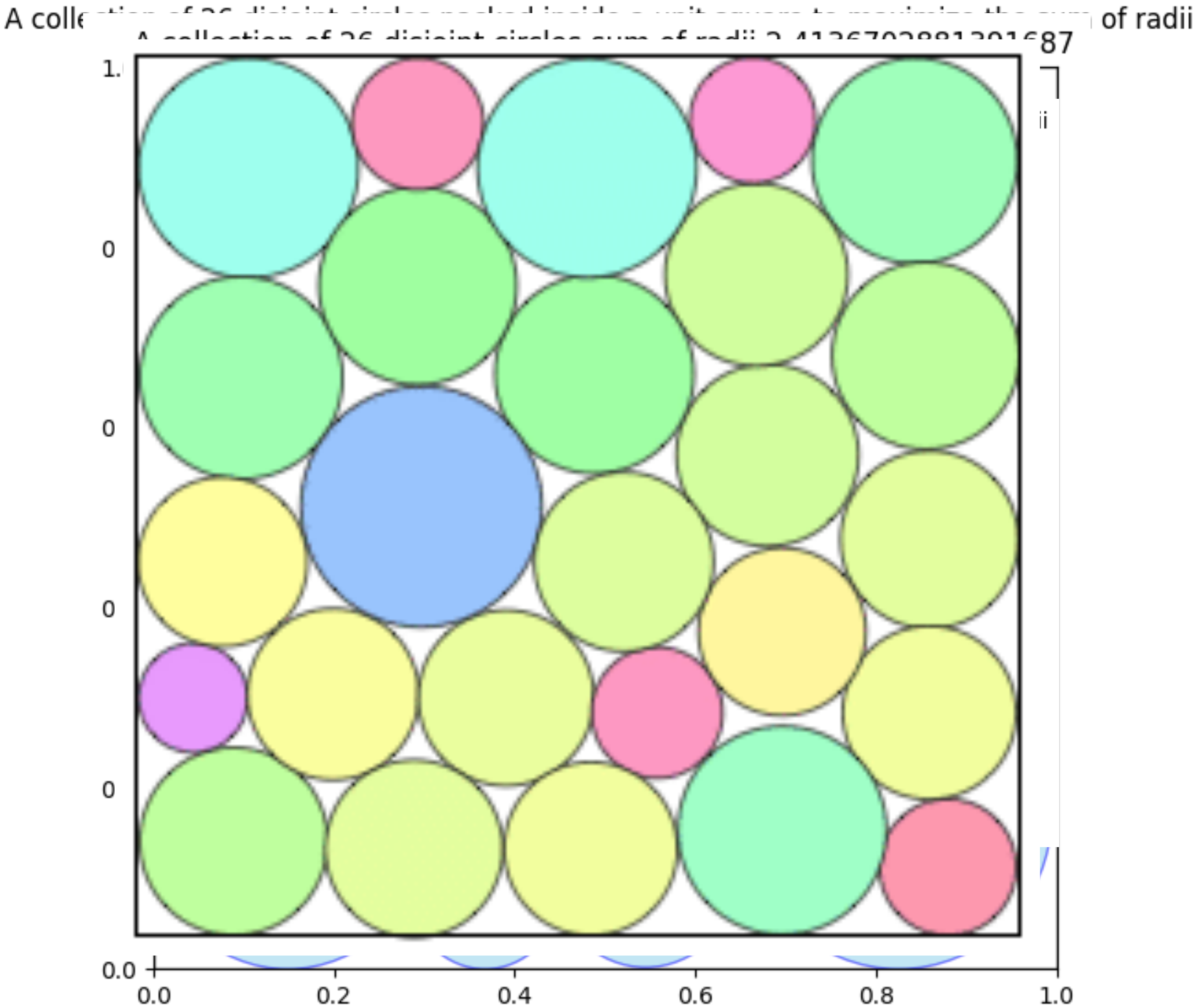

Circle Packing Problem

Previous Best	2.634+

From MIQCP to MINLP

Market impact

- Start from previous portfolio model
- Large trades have a disproportionate impact
- Measure through a nonlinear term
  - Example: $x^{1.5}$

```
m.setObjective(μ @ x, gp.GRB.MAXIMIZE)
m.addConstr(x @ Σ @ x <= max_risk)
m.addConstr(z.sum() <= K)

b = m.addVar(lb=1.0, ub=1.0)
m.addConstr(b == x.sum()
                  + 0.01 * (x ** (1.5)).sum())
```

Limitations of Nonlinear Modeling APIs

- Nonlinear constraints expressed as
  $$y = f(x)$$
- Python provides natural algebraic modeling
- Other APIs require explicit expression trees
- Not available in MATLAB or R

```
m.setObjective(μ @ x, gp.GRB.MAXIMIZE)
m.addConstr(x @ Σ @ x <= max_risk)
m.addConstr(z.sum() <= K)

b = m.addVar(lb=1.0, ub=1.0)
m.addConstr(b == x.sum()
            + 0.01 * (x ** (1.5)).sum())
```

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Solving Portfolio with Market Impact

```
Optimize a model with 463 rows, 925 columns and 1386 nonzeros (Max)
Model fingerprint: 0x6c4fff96
Model has 462 linear objective coefficients
Model has 1 quadratic constraint
Model has 1 general nonlinear constraint (462 nonlinear terms)
Variable types: 463 continuous, 462 integer (462 binary)
```

```
H    0     0                          0.3171485    0.34635  9.21%     -    8s
     0     2    0.34635    0   11      0.31715    0.34635  9.21%     -    8s
H  122   131                          0.3171485    0.34616  9.15%  10.7    9s
```

```
 16919 12377    0.31809  173    3    0.31807    0.31810  0.01%  11.2  110s

Explored 18332 nodes (202563 simplex iterations) in 111.70 seconds (124.07
work units)
Thread count was 11 (of 11 available processors)

Solution count 10: 0.318071 0.318071 0.318071 ... 0.317215

Optimal solution found (tolerance 1.00e-04)
Best objective 3.180712054888e-01, best bound 3.181029805467e-01, gap 0.0100%
```

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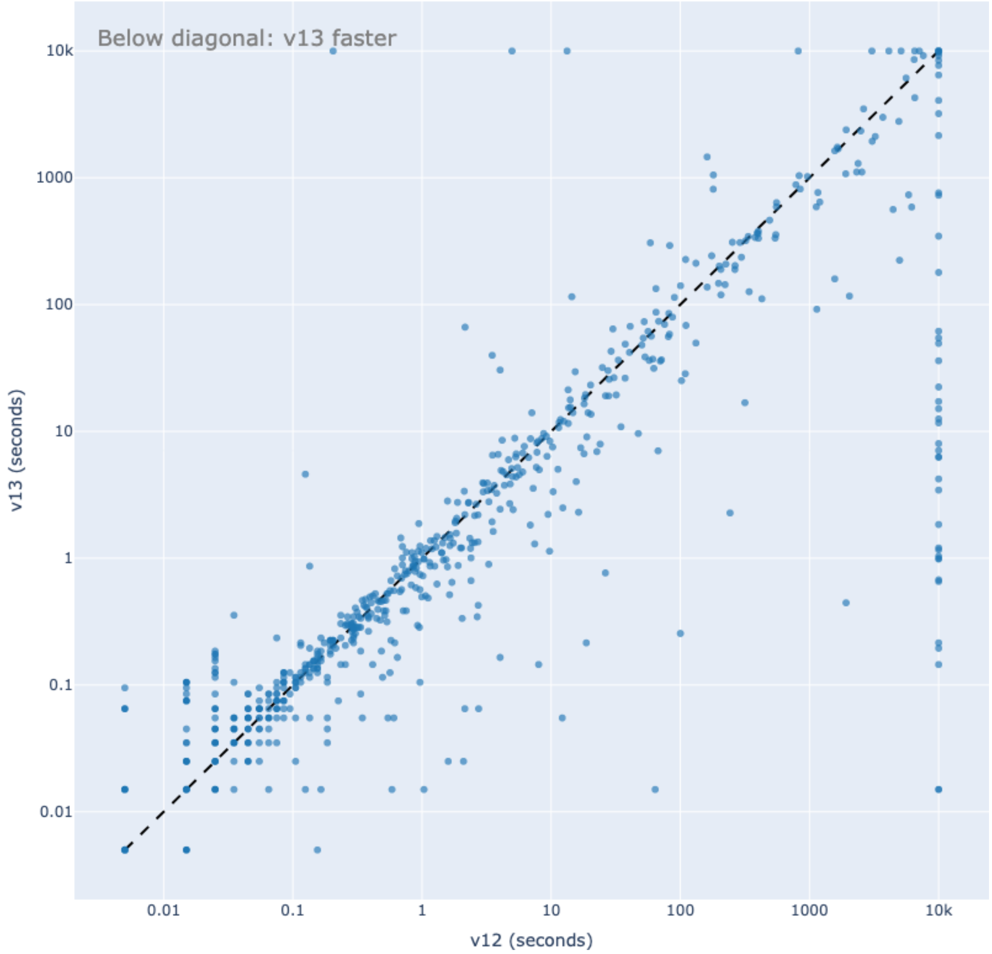
Improving MINLP Performance in v13

All models (874)	> 1s models (329)	> 10s models (203)
1.33 x	2.1 x	3 x



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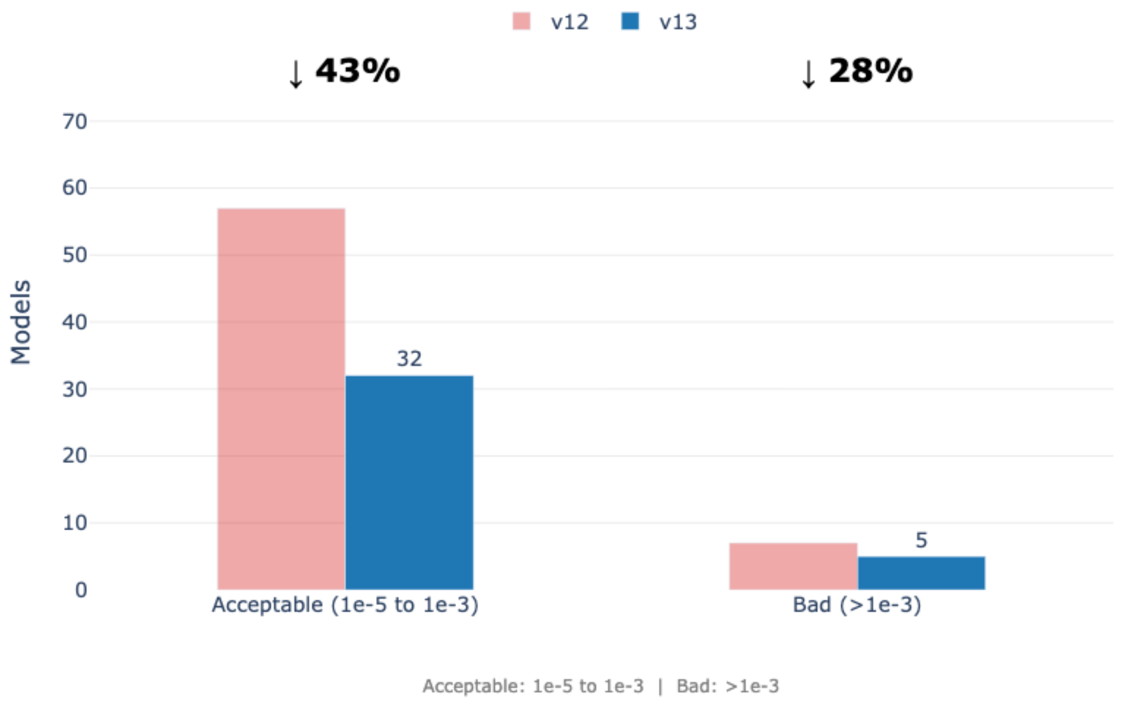
MINLP Solution Quality Improvements

- 1328 models (internal benchmark)
- More solutions found:
  - 1121 → 1187
- Large constraint violations are less frequent



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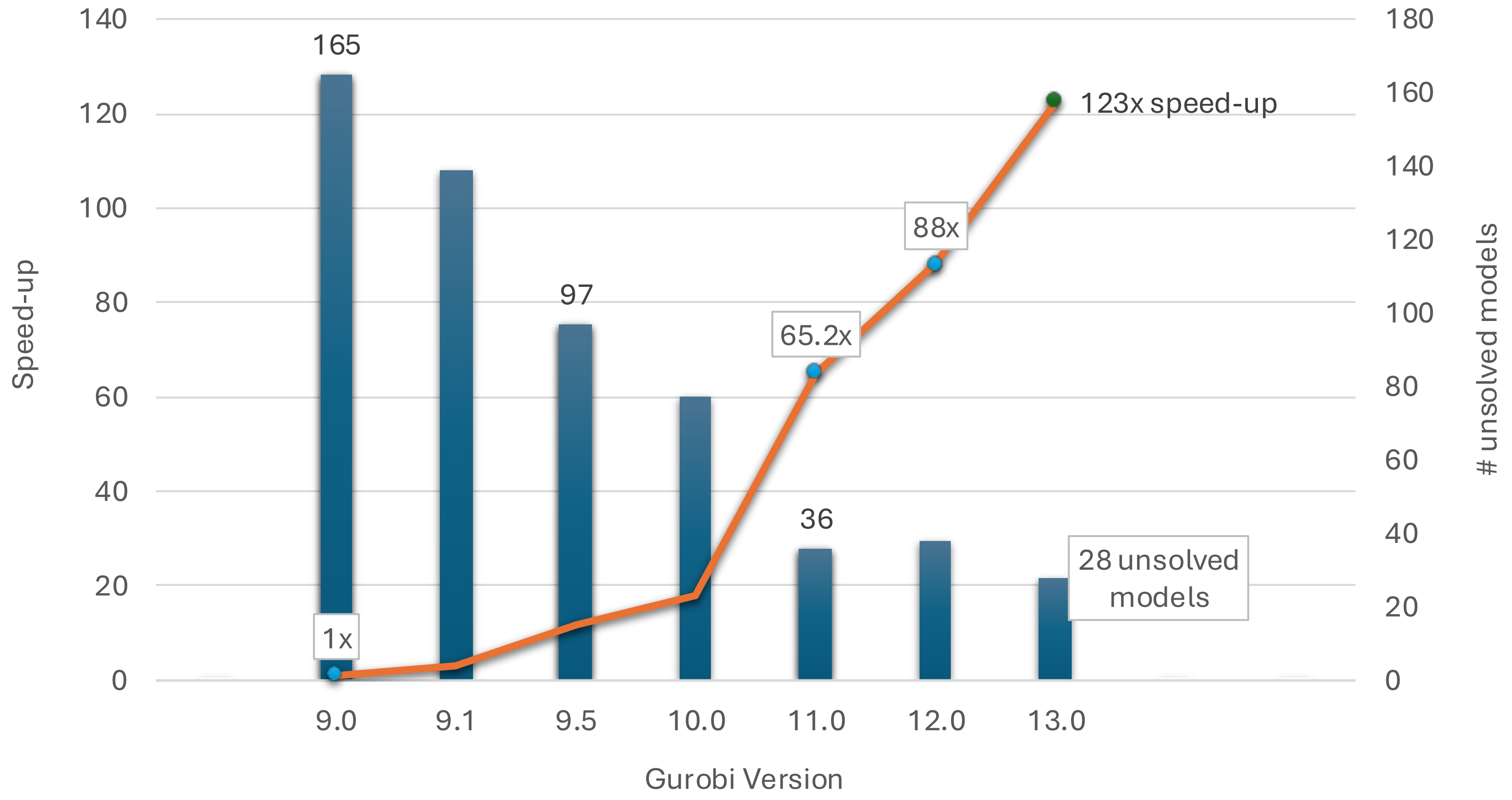
Improving Nonconvex MIQCP Performance



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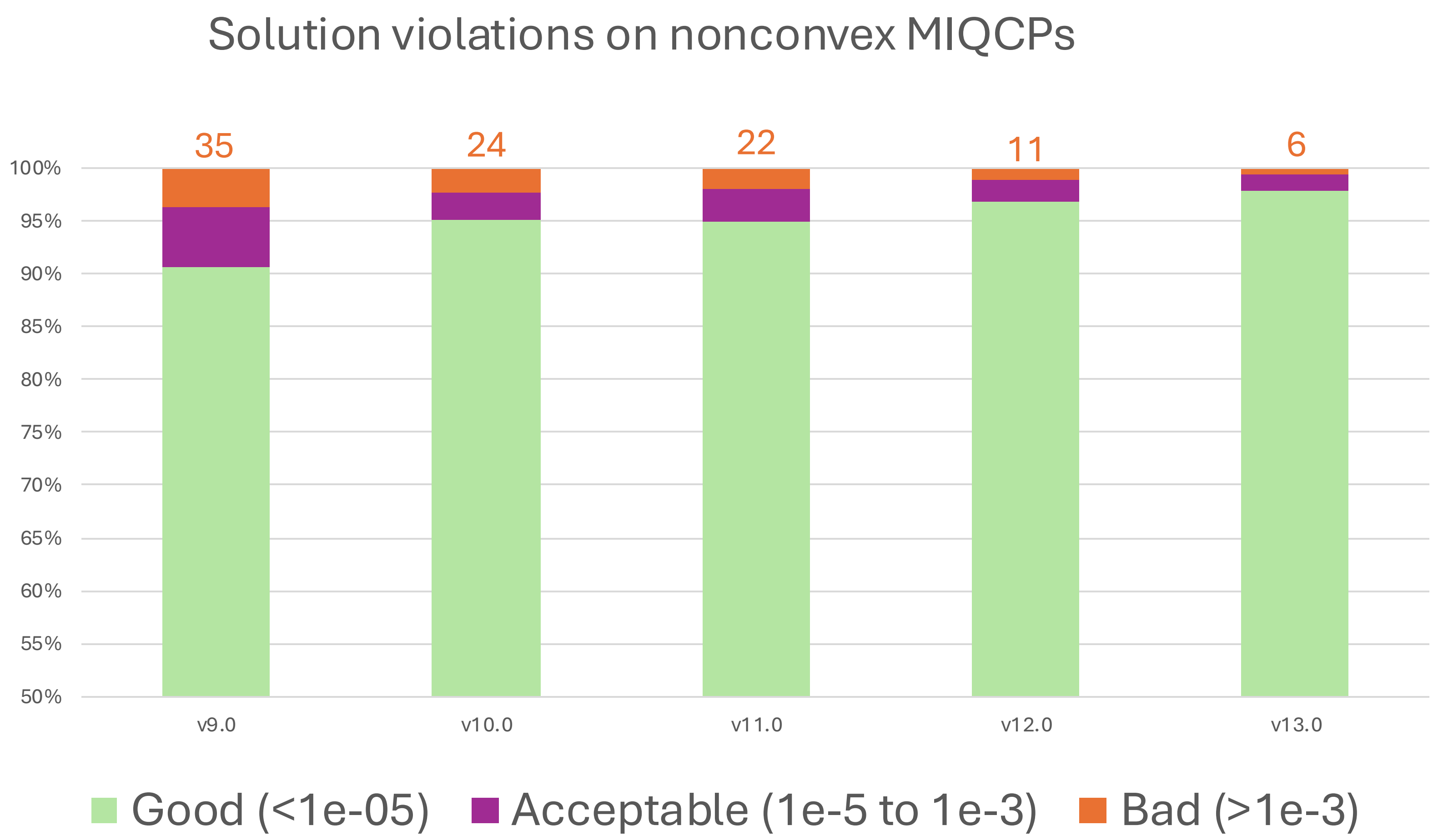
Improving Nonconvex MIQCP Solution Quality



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Conclusions & Outlook

Conclusions

Convex MIQP, MIQCP

- Mature algorithms
- Reliable at scale
- Second-order cone modeling is key

Nonconvex MINLP

- Strong performance for quadratic nonconvex
- Foundations still developing
- Rapid progress on general MINLP

Continuous nonlinear models

- Nonlinear Barrier: powerful option
  - Scales to very large models
  - High accuracy local solutions

Outlook

Usability

- More natural modeling
- In more languages

Solution quality

- Continued focus on robustness and accuracy

Performance

- Tighter integration with Nonlinear Barrier
- Better exploitation of convexity in nonlinear functions

Thank you!